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ON A QUESTION OF SERGEI AKBAROV

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Abstract. Let G be a topological group, let $\pi_1, \pi_2, \dots, \pi_n$ be pairwise inequivalent bounded irreducible representations of G in finite-dimensional complex vector spaces $E_1, E_2, \dots, E_n$, respectively, and let $M(G)$ be the measure algebra of G. Denote by the same symbols π_i, $i = 1, 2, \dots, n$, the extensions of the group representations π_i of G to representations of the measure algebra $M(G)$. It is proved that in this case, $\pi(M(G)) = \dot{+}_{i=1}^{n} B(E_i)$.

§ 1. Introduction

Akbarov [1] posed the following question. Let K be a compact group and let $\pi_1 \colon K \to B(H_1)$ and $\pi_2 \colon K \to B(H_2)$ be two inequivalent irreducible unitary representations of K. These representations can be extended to homomorphisms of the measure algebra $\varphi_1 \colon M(K) \to B(H_1)$ and $\varphi_2 \colon M(K) \to B(H_2)$, where $B(H)$ stands for the algebra of all linear operators in a finite-dimensional vector space H.

The question is as follows: let $A_1 \in B(H_1)$ and $A_2 \in B(H_2)$ be two operators. Is there an element $\mu \in M(K)$ such that $\varphi_1(\mu) = A_1$ and $\varphi_2(\mu) = A_2$?

The objective of this note is to answer a more general question in the affirmative.

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§ 2. Preliminaries

Lemma 1. *Let G be a topological group, let π be a bounded irreducible representation of G in a finite-dimensional complex vector space E, and let $M(G)$ be the measure algebra of G. Denote by the same symbol π the extension of the group representation π of G to a representation of the measure algebra $M(G)$. Then $\pi(M(G)) = B(E)$.*

Proof. This follows immediately from Burnside's theorem (for the shortest proof, see [2]).

Corollary. *Under the assumptions of Lemma 1, every operator $A \in B(E)$ is the image of some element $\mu \in M(K)$.*

Lemma 2. *Let G be a locally compact group and let $L^1(G)$ be the group algebra of G with respect to the left-invariant Haar measure. Let π be a continuous bounded irreducible representation of G in a finite-dimensional complex vector space E. Denote by the same symbol π the extension of the group representation π of G to a representation of $L^1(K)$. Then $\pi(L^1(G)) = B(E)$.*

Proof. This follows also immediately from Burnside's theorem.

Let us recall the following general result concerning finite-dimensional representations of algebras over an arbitrary field.

Lemma 3 ([3], Proposition 3.1.4)**.** *Let π_i, $1 \le i \le m$, be irreducible finite-dimensional pairwise nonequivalent representations of an algebra A, and let ρ be a subrepresentation of the direct sum*

$$\pi = \dot{+}_{i=1}^{m} n_i \pi_i.$$

Then ρ is isomorphic to $\dot{+}_{i=1}^{m} r_i \pi_i$, $r_i \le n_i$, and the inclusion $\varphi\colon \rho \to \pi$ is a direct sum of inclusions

$$\varphi_i\colon r_i\pi_i \to n_i\pi_i$$

given by multiplication of a row vector of elements of π_i (of length r_i) by a certain $r_i \times n_i$ matrix X_i with linearly independent rows:

$$\varphi(v_1, \dots, v_{r_i}) = (v_1, \dots, v_{r_i}) X_i, \qquad i = 1, \dots, m.$$

§ 3. Main theorem

Denote the extension of every group representation ρ of G to a representation of the measure algebra $M(G)$ by the same symbol ρ.

Theorem 1. *Let G be a topological group, let $\pi_1, \pi_2, \dots, \pi_n$ be pairwise inequivalent continuous bounded irreducible representations of G in finite-dimensional complex vector spaces $E_1, E_2, \dots, E_n$, respectively, and let $M(G)$ be the measure algebra of G. Let π be the direct sum $\dot{+}_{i=1}^n \pi_i$. Then*

$$\pi(M(G)) = \dot{+}_{i=1}^n B(E_i).$$

Proof. Since the representations π_i, $i = 1, \dots, n$, are bounded and continuous, it follows that each of these representations admits an extension to $M(G)$, and the representation π also admits an extension to $M(G)$. Let us apply Lemma 3 with $A = M(G)$.

It is clear that the representation of A extended from π is a subrepresentation of the direct sum of the representations of A extended from π, which follows immediately from Schur's lemma (say, in the form presented in Proposition 2.3.9 of [3]), because the ith diagonal block of $\pi(a)$, $i = 1, \dots, m$, turns out to be equal to $\pi_i(a)$ for every $a \in A$ provided that this holds for $a = g \in G$.

It follows from Lemma 3 that the above two representations of A in the direct sum of the representation spaces E_i of pi_i, $i = 1, \dots, n$, coincide.

This implies that

$$(\dot{+}_{i=1}^n \pi_i)(A) = \dot{+}_{i=1}^n \pi_i(A).$$

Recall that π_i, $i = 1, \dots, n$, are nonequivalent irreducible representations of A. Let $I_i = \ker \pi_i$ be the two-sided ideal of A defined by the kernel of π_i, $i = 1, \dots, n$. Consider the restriction of π_2 to I_1. Assume that this restriction is zero. Then both π_1 and π_2 define irreducible representations of the simple quotient algebra A/I_1, and hence these representations are equivalent (Corollary 4.6 of Chap. XVII, §4 in [4]), which contradicts the assumption of the theorem. By induction, this consideration can be extended to any finite family of pairwise inequivalent irreducible representations. In particular, the restriction of π_1 to the two-sided ideal $\bigcap_{i=2}^n I_i$ is nonzero. Then the essential subspace of this restriction is a nonzero invariant subspace of E_1. Since π_1 is irreducible, it follows that the essential subspace is E_1. We claim now that the restriction $\pi_1|_{\bigcap_{i=2}^n I_i}$ is obviously irreducible, because every subspace of E_1 invariant with respect to $\bigcap_{i=2}^n I_i$ is invariant with respect to A. Applying Burnside's theorem to the algebra $\bigcap_{i=2}^{n} I_i$ and the representation $\pi_1|_{\bigcap_{i=2}^{n} I_i}$, we see that

$$\pi_1(\bigcap_{i=2}^{n} I_i) = B(E_1),$$

and similar equations hold for $\pi_2, \ldots, \pi_n$. Thus,

$$\pi(A) = (\dot{+}_{i=1}^{n} \pi_i)(A) = \dot{+}_{i=1}^{n} B(E_i),$$

which completes the proof of the theorem.

Denote the extension of every group representation ρ of G to a representation of the measure algebra $L^1(G)$ by the same symbol ρ.

Theorem 2. *Let G be a locally compact group, let $\pi_1, \pi_2, \ldots, \pi_n$ be pairwise inequivalent continuous bounded irreducible representations of G in finite-dimensional complex vector spaces $E_1, E_2, \ldots, E_n$, respectively, and let $L^1(G)$ be the measure algebra of G. Let π be the direct sum $\dot{+}_{i=1}^{n} \pi_i$. Then*

$$\pi(L^1(G)) = \dot{+}_{i=1}^{n} B(E_i).$$

Proof. The proof almost repeats that of Theorem 1; only the reference to Lemma 1 is to be replaced by that to Lemma 2.

§ 4. Questions

In connection with the above facts concerning the finite-dimensional representations (which are especially important for continuous irreducible unitary representations of Moore groups, i.e., the locally compact groups all of whose continuous unitary irreducible representations are finite-dimensional), it is of interest to find out whether or not there are non-Moore locally compact groups of type I (and even groups all of whose irreducible representations define representations of the group algebra by compact operators in the representation space) whose group algebra representations defined by infinite-dimensional representations of the group take the group algebra onto the algebra of compact operators in the corresponding representation space (this property is an infinite-dimensional analog of the assertion of Lemma 2).

If this class of groups is meaningful, then it would be of interest to find an analog of the theorem, using the involution in the group algebra and the corresponding apparatus of irreducible representations of algebras with involution [5].

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